Algebraic Equations

Q1. *[Warm-up.]* Find all the solutions to the following equations.

(a) $x^2 - 9 = 0$.

(b) $x^2 - x - 2 = 0$.

(c) $x^2 + ax + a^2$, where a is a constant. (Express your solution in terms of a.)

(d) $x^3 - a^3 = 0$, where a is a constant. [**Hint:** What do you get if you multiply out $(x-a)(x^2+ax+a^2)$?]

Solution:

(a) We can take square roots of $x^2 = 9$ but let's be lazy and apply the quadratic formula! We get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 + \sqrt{0^2 + (4)(9)}}{2} = \frac{\pm 6}{2} = \pm 3.$$

(b) We could try to guess the factors, but let's apply the quadratic formula again:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(-2)}}{2} = \frac{1 \pm 3}{2}.$$

So the roots are $x = 2$ and $x = -1$.

(c) Probably best to just apply the quadratic formula here! We get

$$x = \frac{-a \pm \sqrt{a^2 - 4a^2}}{2} = \frac{-a \pm \sqrt{-3a^2}}{2}.$$

The $\sqrt{-3a^2}$ is a bit troubling, but let's ignore that for now...

(d) If we multiply out $(x-a)(x^2+ax+a^2)$ we get

$$x^3 + ax^2 + a^2x - ax^2 - a^2x - a^3 = x^3 - a^3.$$

So the roots of

$$x^3 - a^3 = 0$$

are the same as the roots of

$$(x-a)(x^2+ax+a^2) = 0.$$

This is an easier problem to solve! One root is $x = a$, and the others are the roots of the quadratic equation

$$x^2 + ax + a^2 = 0$$

that we solved in part (a). So the roots of $x^3 - a^3 = 0$ are

$$a, \quad \frac{-a+\sqrt{-3a^2}}{2}, \quad \text{and} \quad \frac{-a-\sqrt{-3a^2}}{2}.$$

Note: If we re-write the equation as $x^3 = a^3$ and take cube roots, we immediately get the root $x = a$. But here we discovered two additional roots. In general, a cubic equation will have three roots.

Q2. *[A derivation of the quadratic formula.]* Consider the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

(a) Show that if we substitute $x = X + d$ into the equation then expand and simplify, we can turn it into

$$aX^2 + (2ad + b)X + (ad^2 + bd + c) = 0.$$

(b) By making an appropriate choice of d (which will depend on some of a, b, c), show that the equation in part (a) can be written in the form

$$aX^2 + p = 0.$$

Your p should be written in terms of a, b and c. [**Hint:** There is no X term in this equation. So try to get rid of the X term in (a)!]

(c) Solve your equation in part (b) for X, and therefore solve the original quadratic equation for x.

Solution:

(a) The left-side becomes

$$\begin{aligned} a(X+d)^2 + b(X+d) + c &= a(X^2 + 2dX + d^2) + bX + bd + c \\ &= aX^2 + (2ad + b)X + (ad^2 + bd + c). \end{aligned}$$

(b) We want the coefficient of X to be zero:

$$2ad + b = 0.$$

We can do this by choosing $d = -b/(2a)$. This leaves us with

$$aX^2 + (0)X + \left[a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c\right] = 0,$$

which we can simplify to

$$aX^2 + \frac{-b^2 + 4ac}{4a} = 0.$$

(c) The above equation can be re-written as

$$X^2 = \frac{b^2 - 4ac}{4a^2}$$

so its solutions are given by

$$X = \pm\sqrt{b^2 - 4ac}2a.$$

Hence

$$x = X + d = X + \frac{-b}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(As expected!!)

Q3. *[Vieta's relations.]*

(a) Let r_1 and r_2 be the roots of the quadratic equation $ax^2+bx+c=0$. Prove the following expressions for the sum and product of the roots:

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 r_2 = \frac{c}{a}.$$

[*Challenge:* Can you give two different proofs? Can you give more than two?]

(b) Vieta's relations tell you that you can "reconstruct" a quadratic equation if you know its roots. For example, can you find a quadratic equation whose roots are 1 and -3? [**Hint:** To make your life easier, choose a quadratic equation where $a = 1$, i.e., an equation of the form $x^2 + bx + c = 0$.]

(c) Let $r_1,\ r_2$ and r_3 be the roots of the cubic equation $ax^3+bx^2+cx+d=0$. Discover and prove a relationship between the roots $r_1,\ r_2$ and r_3 and the coefficients $a,\ b,\ c$ and d of the cubic, similar to what we have in part (a). [**Hint:** Expand $a(x-r_1)(x-r_2)(x-r_3)$.]

Solution:

(a) **Proof 1:** If the roots are r_1 and r_2, then we can factor the polynomial as

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Expanding the right-side, we arrive at

$$ax^2 + bx + c = a(x^2 - (r_1 + r_2)x + r_1 r_2) = ax^2 - a(r_1 + r_2)x + ar_1 r_2.$$

Now compare the constant terms and coefficients of x on both sides of the equation above to get

$$b = -a(r_1 + r_2) \quad \text{and} \quad c = ar_1 r_2.$$

The desired expressions follow by diving through by a.

Proof 2: Using the quadratic formula, we can let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}.$$

Similarly,

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

(b) Let's try to find an equation of the form $x^2 + bx + c = 0$ whose roots are $r_1 = 1$ and $r_2 = -3$. From Vieta's relations, we see that coefficient of x will be the (-1) times sum of the roots: $b = -(1 + (-3)) = 2$.
The constant term will be the product of the roots: $c = (1)(-3) = -3$.
So our desired quadratic equation is

$$x^2 + 2x - 3 = 0.$$

[*Note:* You should confirm, e.g. by using the quadratic formula, that the roots of this equation are indeed 1 and -3!]

(c) We can repeat the same kind of thing that we did in **Proof 1** in part (a):

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x - r_1)(x - r_2)(x - r_3) \\ &= \cdots \qquad \text{(expand!)} \\ &= ax^3 - a(r_1 + r_2 + r_3)x^2 + a(r_1r_2 + r_1r_3 + r_2r_3)x - ar_1r_2r_3. \end{aligned}$$

If we compare the coefficients of x^2, we get

$$r_1 + r_2 + r_3 = -\frac{b}{a}$$

and if we compare the constant terms, we get

$$r_1r_2r_3 = -\frac{c}{a}.$$

(We also get another interesting relation from the coefficients of x.)

Q4. *[Solving the depressed cubic.]* Consider the cubic equation

$$x^3 + px + q = 0. \qquad (*)$$

(In particular, there is no x^2 term: it's been "depressed.")

(a) Let r be a root of $(*)$. For the moment, express r in the form $r = \alpha + \beta$, where α and β are unknown variables. Show that

$$\alpha^3 + \beta^3 + (\alpha + \beta)(3\alpha\beta + p) + q = 0.$$

(b) Show further that, if $\alpha\beta = -p/3$, then

$$\alpha^3\beta^3 = -\frac{p^3}{27} \quad \text{and} \quad \alpha^3 + \beta^3 = -q.$$

Using this and Vieta's relations from the previous problem, find a quadratic equation whose roots are α^3 and β^3.

(c) By solving your quadratic equation in part (b), find expressions for α and β, and hence for the root r. [*Note:* This gives you **Cardano's formula**.]

Solution:

(a) If we plug $r = \alpha + \beta$ into $(*)$ then expand and simplify, we arrive at the given equation.

(b) If $\alpha\beta = -p/3$, then by raising both sides to the power of 3, we get

$$\alpha^3\beta^3 = -\frac{p^3}{27}.$$

Next, we cna re-arrange $\alpha\beta = -p/3$ into $3\alpha\beta + p = 0$. If we plug this into the equation in (a), we get

$$\alpha^3 + \beta^3 + q = 0$$

hence

$$\alpha^3 + \beta^3 = -q.$$

So now, if we pretend that α^3 and β^3 are solutions to a quadratic equation of the form

$$y^2 + By + C = 0$$

then from Vieta's relations we get that

$$B = -(\alpha^3 + \beta^3) = q \quad \text{and} \quad C = \alpha^3\beta^3 = -\frac{p^3}{27}.$$

That is, our quadratic equation will look like

$$y^2 + qy - \frac{p^3}{27} = 0.$$

(c) If we apply the quadratic formula to the quadratic equation in (b), we get

$$y = \frac{-q \pm \sqrt{q^2 + \frac{4}{27}p^3}}{2}.$$

But we know that the roots of the quadratic equation in (b) are α^3 and β^3! So we have

$$\alpha^3 = \frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}$$

and

$$\beta^3 = \frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}.$$

(We assigned the $\pm$ signs at random.)
Taking cube roots, we get

$$\alpha = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and

$$\beta = \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

(There is a subtlety here involving take roots of numbers!)

Thus,

$$r = \alpha + \beta = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

(But a cubic equation has three roots, right? How do we get the other two? This is related to the subtlety mentioned above.)

Q5. Consider the depressed cubic

$$x^3 + x - 2 = 0.$$

(a) Show that $x = 1$ is a root of the equation.

(b) Apply Cardano's formula from **Q4** to this equation. Do you get the root 1?
You might want to plug your formula into a calculator, such as Wolframalpha.
[*Note:* To enter square roots and cube roots into Wolframalpha, use `sqrt()` and `cbrt()`. So, for example, enter `sqrt(2) for` $\sqrt{2}$ `and cbrt(10) for` $\sqrt[3]{10}$.]

Solution:

(a) Simply plug in $x = 1$ and notice that

$$x^3 + x - 2 = 1 + 1 - 2 = 0.$$

(b) Cardano's formula says that

$$\begin{aligned} r &= \sqrt[3]{\frac{2}{2} + \sqrt{\frac{(-2)^2}{4} + \frac{1^3}{27}}} + \sqrt[3]{\frac{2}{2} + \sqrt{\frac{(-2)^2}{4} + \frac{1^3}{27}}} \\ &= \sqrt[3]{1 + \sqrt{\frac{28}{27}}} + \sqrt[3]{1 - \sqrt{\frac{28}{27}}} \end{aligned}$$

is a root of this equation. Hmm...

If we plug this expression into Wolframalpha, say, we see that it's equal to 1! Curious. Can you prove this algebraically?

Note: I'm being deliberately sneaky and misleading here. As mentioned in the solution to the previous problem, there's something subtle going on with the cube roots. Kind of like how we need $\pm$ next to the square roots in the quadratic formula, we need some kind of adjustment in the cubic formula to account for all the possible roots.

Q6. *[Solving the general cubic.]* Consider the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

By dividing through by a, we can (and will) assume that $a = 1$, so our equation takes the simpler form

$$x^3 + bx^2 + cx + d = 0. \quad (**)$$

(a) Show that if we substitute $x = X + e$ into equation $(**)$ and simplify we can re-write the equation as

$$X^3 + (3e + b)X^2 + (3e^2 + 2be + c)X + (e^3 + be^2 + ce + d) = 0.$$

(b) Pick a choice of e that turns your equation in part (a) into a depressed cubic.

(c) Use Cardano's formula, from **Q4**, to solve $(**)$.

Solution:

(a) Let's look at the left-side. We have:

$$\begin{aligned} x^3 + bx^2 + cx + d &= (X + e)^3 + b(X + e)^2 + c(X + e) + d \\ &= (X^3 + 3X^2e + 3Xe^2 + e^3) + b(X^2 + 2Xe + e^2) + c(X + e) + d \\ &= X^3 + (3e + b)X^2 + (3e^2 + 2be + c)X + (e^3 + be^2 + ce + d). \end{aligned}$$

This is exactly what we wanted to get.

(b) We want the coefficient of X^2 to be zero, that is, we want

$$3e + b = 0.$$

We can accomplish this by choosing $e = -b/3$. The resulting depressed cubic is then

$$X^3 + \left(c - \frac{b^2}{3}\right) X + \left(\frac{4b^3}{27} + \frac{bc}{3} + d\right) = 0.$$

(c) This is a matter of plugging

$$p = c - \frac{b^2}{3} \quad \text{and} \quad q = \frac{4b^3}{27} + \frac{bc}{3} + d$$

into Cardano's formula to solve for X. We can then get x as $x = X - b/3$. The result is not pretty (and therefore I'll omit it), but it is what it is!